

THE PROJECTIVE INDECOMPOSABLE MODULES FOR THE RESTRICTED ZASSENHAUS ALGEBRAS IN CHARACTERISTIC 2

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ABSTRACT. It is shown that for the restricted Zassenhaus algebra $\mathfrak{W} = \mathfrak{W}(1; n)$, $n > 1$, defined over an algebraically closed field \mathbb{F} of characteristic 2 any projective indecomposable restricted \mathfrak{W} -module has maximal possible dimension 2^{2^n-1} , and thus is isomorphic to some induced module $\text{ind}_{\mathfrak{t}}^{\mathfrak{W}}(\mathbb{F}(\mu))$ for some torus of maximal dimension \mathfrak{t} . This phenomenon is in contrast to the behavior of finite-dimensional non-solvable restricted Lie algebras in characteristic $p > 3$ (cf. [4, Theorem 6.3]).

1. INTRODUCTION

Let \mathfrak{L} be a finite-dimensional restricted Lie algebra defined over an algebraically closed field \mathbb{F} of characteristic $p > 0$, and let $\mathfrak{t} \subseteq \mathfrak{L}$ be a torus of maximal dimension. Then, as $\underline{u}(\mathfrak{t})$ - the *restricted universal enveloping algebra* of \mathfrak{t} - is a commutative and semi-simple associative \mathbb{F} -algebra, every irreducible restricted \mathfrak{t} -module is 1-dimensional and also projective. Moreover, there exists a canonical one-to-one correspondence between the isomorphism classes of irreducible restricted \mathfrak{t} -modules and the set

$$(1.1) \quad \mathfrak{t}^{\otimes} = \text{span}_{\mathbb{F}_p} \{ t \in \mathfrak{t} \mid t^{[p]} = t \}^*,$$

where $\mathbb{F}_p \subseteq \mathbb{F}$ denotes the prime field, and $\underline{}^* = \text{Hom}_{\mathbb{F}_p}(\underline{}, \mathbb{F}_p)$. For $\mu \in \mathfrak{t}^{\otimes}$ let $\mathbb{F}(\mu)$ denote the corresponding irreducible restricted \mathfrak{t} -module. Then

$$(1.2) \quad P(\mu) = \text{ind}_{\mathfrak{t}}^{\mathfrak{L}}(\mathbb{F}(\mu)) = \underline{u}(\mathfrak{L}) \otimes_{\underline{u}(\mathfrak{t})} \mathbb{F}(\mu)$$

is a projective restricted \mathfrak{L} -module (cf. [8, Proposition 2.3.10]), but in general it will be decomposable. Indeed, it has been shown in [4, Theorem 6.3] that for $p > 3$ the finite-dimensional restricted Lie algebra \mathfrak{L} is solvable if, and only if, $P(0)$ is indecomposable. The authors conclude their paper with the remark, that for $p = 2$ the “if” part of the assertion is false, e.g., for $p = 2$ and \mathfrak{L} equal to the restricted Zassenhaus algebra $\mathfrak{W}(1; 2)$ one has that $P(0)$ is indecomposable and coincides with the projective cover of the trivial $\mathfrak{W}(1; 2)$ -module. However, $\mathfrak{W}(1; 2)$ is a non-solvable restricted Lie algebra. The main purpose of this note is to extend this result to all restricted Zassenhaus algebras $\mathfrak{W}(1; n)$, $n > 1$, in characteristic 2 and to all projective restricted modules $P(\mu)$, $\mu \in \mathfrak{t}^{\otimes}$.

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Theorem. *Let $n \geq 1$, let $\mathfrak{W}(1; n)$ be a restricted Zassenhaus algebra defined over an algebraically closed field of characteristic 2, and let $\mathfrak{t} \subseteq \mathfrak{W}(1; n)$ be a torus of maximal dimension. Then $P(\mu)$ is indecomposable for all $\mu \in \mathfrak{t}^{\otimes}$. In particular, all projective indecomposable restricted $\mathfrak{W}(1; n)$ -modules have maximal possible dimension 2^{2^n-1} , and $\mathfrak{W}(1; n)$ has maximal 0-p.i.m.*

The proof of the theorem will be arranged in two major steps. First, it will be shown that every restricted Zassenhaus algebra $\mathfrak{W}(1; n)$ has a direct sum decomposition

$$(1.3) \quad \mathfrak{W}(1; n) = \mathfrak{t}_0 \oplus \mathfrak{B},$$

where \mathfrak{t}_0 is a torus of maximal dimension, and \mathfrak{B} is a certain restricted Lie subalgebra of $\mathfrak{W}(1; n)$. In the second step it will be shown that the trivial $\mathfrak{W}(1; n)$ -module \mathbb{F} and $L = W(1; n)^{(1)}$ are the only isomorphism types of irreducible restricted $\mathfrak{W}(1; n)$ -modules. From this fact and [4, Lemma 6.1] one concludes that $\mathfrak{W}(1; n)$ has *maximal 0-p.i.m.*, i.e., the projective cover $P_{\mathbb{F}}$ of the trivial $\mathfrak{W}(1; n)$ -module has maximal possible dimension 2^{2^n-1} , and thus must be isomorphic to $P(0)$ (cf. (1.2)). An elementary calculation then shows that the projective cover P_L of the restricted $\mathfrak{W}(1; n)$ -module L must be isomorphic to $P(\mu)$ for $\mu \in \mathfrak{t}^{\otimes} \setminus \{0\}$, i.e., any projective indecomposable restricted $\mathfrak{W}(1; n)$ -module has the maximal possible dimension 2^{2^n-1} (cf. Remark 3.7). As a by-product we may also conclude that the restricted Lie subalgebra \mathfrak{B} has properties similar to the Borel subalgebra in a classical restricted Lie algebra (cf. Remark 2.2 and Theorem 3.5(b)).

2. PRELIMINARIES

From now on we will assume that \mathbb{F} is a field of characteristic $p > 0$.

2.1. Simple algebras and simple restricted Lie algebras. A (restricted) \mathbb{F} -Lie algebra \mathfrak{L} is said to be *simple* if any (restricted) Lie ideal \mathfrak{J} of \mathfrak{L} coincides either with \mathfrak{L} or with 0. The following fact is straightforward and its easy proof is left to the reader (cf. [5, §6]).

Fact 2.1. *Let \mathbb{F} be a field of characteristic $p > 0$.*

- (a) *Let \mathfrak{L} be a non-abelian finite-dimensional simple restricted \mathbb{F} -Lie algebra, and let $L = \text{soc}_{\mathfrak{L}}(\mathfrak{L})$ denote the socle of the adjoint \mathfrak{L} -module. Then L is a minimal Lie ideal of \mathfrak{L} and a non-abelian simple \mathbb{F} -Lie algebra. Moreover, \mathfrak{L} coincides with the minimal p -envelope $\text{env}_p(L)$ of L .*
- (b) *Let L be a non-abelian finite-dimensional simple \mathbb{F} -Lie algebra. Then the minimal p -envelope $\text{env}_p(L)$ is a finite-dimensional simple restricted \mathbb{F} -Lie algebra, and one has $L = \text{soc}_{\text{env}_p(L)}(\text{env}_p(L))$.*

2.2. Generalized Borel subalgebras. Let \mathfrak{L} be a finite-dimensional restricted Lie algebra over an algebraically closed field \mathbb{F} of characteristic $p > 0$. By $\text{Irr}_p(\mathfrak{L})$ we will denote the set of isomorphism classes of restricted irreducible \mathfrak{L} -modules. A proper restricted Lie subalgebra \mathfrak{B} of \mathfrak{L} will be said to be a *generalized Borel subalgebra*, if

- (i) the unipotent radical $\text{rad}_u(\mathfrak{B})$ of \mathfrak{B} is non-trivial, i.e., $\text{rad}_u(\mathfrak{B}) \neq 0$;
- (ii) for any irreducible restricted \mathfrak{L} -module S , $[S] \in \text{Irr}_p(\mathfrak{L})$, the restricted $\mathfrak{B}/\text{rad}_u(\mathfrak{B})$ -module $S^{\text{rad}_u(\mathfrak{B})}$ is irreducible;

- (iii) the map $\chi_{\mathfrak{B}}: \text{Irr}_p(\mathfrak{L}) \rightarrow \text{Irr}_p(\mathfrak{B}/\text{rad}_u(\mathfrak{B}))$ given by $\chi_{\mathfrak{B}}([S]) = [S^{\text{rad}_u(\mathfrak{B})}]$ is injective.

Remark 2.2. Let \mathfrak{L} be a classical finite-dimensional simple restricted Lie algebra, and let \mathfrak{B} be a Borel subalgebra of \mathfrak{L} . It is well known that for any irreducible restricted \mathfrak{L} -module S , $S^{\text{rad}_u(\mathfrak{B})}$ is 1-dimensional. Moreover the mapping $\chi_{\mathfrak{B}}: \text{Irr}_p(\mathfrak{L}) \rightarrow \text{Irr}_p(\mathfrak{B}/\text{rad}_u(\mathfrak{B}))$ is injective. Hence \mathfrak{B} is a generalized Borel subalgebra of \mathfrak{L} .

Remark 2.3. A similar generalization of the concept of Borel subalgebra to \mathbb{Z} -graded restricted Lie algebras, and especially to restricted Lie algebra of Cartan type is given in [9, Theorem 4(a) and (c)].

3. THE RESTRICTED ZASSENHAUS ALGEBRA $\mathfrak{W}(1; n)$

Let \mathbb{F} be a field of characteristic $p > 0$, and let \mathbb{E} be a finite subfield of \mathbb{F} . The Lie algebra

$$(3.1) \quad L(\mathbb{E}) = \text{span}_{\mathbb{F}}\{u_a \mid a \in \mathbb{E}\}$$

with bracket given by $[u_a, u_b] = (b - a)u_{a+b}$, $a, b \in \mathbb{E}$, is called the *Zassenhaus algebra* with respect to \mathbb{E} (cf. [10, p. 47ff.]).

Let $\mathcal{O}(1; n)$ denote the truncated divided power \mathbb{F} -algebra of dimension p^n with basis $\{x^{(k)} \mid 0 \leq k \leq p^n - 1\}$. The *Witt algebra* $W(1; n)$ is the Lie subalgebra of $\text{Der}(\mathcal{O}(1; n))$ of *special derivations*, i.e.,

$$(3.2) \quad W(1; n) = \text{span}_{\mathbb{F}}\{e_j = x^{(j+1)} \cdot \partial \mid -1 \leq j \leq p^n - 2\},$$

where ∂ denotes the derivation defined by

$$(3.3) \quad \partial(x^{(a)}) = \begin{cases} 0, & \text{if } a = 0, \\ x^{(a-1)}, & \text{if } 1 \leq a \leq p^n - 1. \end{cases}$$

In particular, one has

$$(3.4) \quad [e_i, e_j] = \begin{cases} c_{ij} \cdot e_{i+j} & \text{if } -1 \leq i \leq p^n - 2, \\ 0 & \text{otherwise,} \end{cases}$$

where $c_{ij} = \binom{i+j+1}{j} - \binom{i+j+1}{i}$. It is well known that if $\mathbb{E} \subseteq \mathbb{F}$ is the finite field with p^n elements and if \mathbb{F} is perfect, then one has a (non-canonical) isomorphism

$$(3.5) \quad L(\mathbb{E}) \simeq W(1; n)$$

(cf. [1], [5], or also [6, Theorem 7.6.3(1)] for p odd). If $p > 2$, then $W(1; n)$ is a simple Lie algebra (cf. [7, Theorem 4.2.4]); while if $p = 2$ and $n > 1$ then

$$(3.6) \quad W(1; n)^{(1)} = [W(1; n), W(1; n)] = \text{span}_{\mathbb{F}}\{e_j \mid -1 \leq j \leq 2^n - 3\}$$

is simple (cf. [2]). The minimal p -envelope $\mathfrak{W}(1; n) = \text{env}_p(W(1; n))$ of the Zassenhaus algebra $W(1; n)$ coincides with the derivation algebra of $W(1; n)$ (cf. [6, Theorem 7.1.2(2) and Theorem 7.2.2(1)]) and is also called the *restricted Zassenhaus algebra*, i.e., identifying $W(1; n)$ with its image in the Lie algebra $\text{Der}(W(1; n))$ one has

$$(3.7) \quad \mathfrak{W}(1; n) = \bigoplus_{1 \leq k \leq n-1} \mathbb{F} \cdot \partial^{[p]^k} \oplus W(1; n).$$

The following result holds.

Proposition 3.1. *If $p = 2$ and $n > 1$, then $\mathfrak{W}(1; n)$ is isomorphic to the minimal 2-envelope of the simple Lie algebra $W(1; n)^{(1)}$.*

Proof. Put $\mathfrak{W} = \mathfrak{W}(1; n) = \text{Der}(W(1; n))$ and consider the canonical injective map

$$(3.8) \quad \alpha: W(1; n)^{(1)} \xrightarrow{j} W(1; n) \xrightarrow{\text{ad}_W} \mathfrak{W}.$$

As $C_{\mathfrak{W}}(\partial) = \text{span}_{\mathbb{F}}\{\partial^{[2]^j} \mid 0 \leq j \leq n-1\}$, one has that $C_{\mathfrak{W}}(A) = 0$ for $A = \text{im}(\alpha)$. Hence, by Fact 2.1, $\mathfrak{H} = \langle A \rangle_p$ - the restricted Lie subalgebra of \mathfrak{W} generated by A - is a minimal p -envelope of $W(1; n)^{(1)}$ (cf. [6, Theorem 1.1.7]). For simplicity we assume that α is given by inclusion. By construction, $\bigoplus_{1 \leq k \leq n-1} \mathbb{F} \cdot \partial^{[p]^k} \oplus W(1; n)^{(1)}$ is an \mathbb{F} -subspace of \mathfrak{H} . An elementary calculation shows that $e_{2^{n-1}-1}^{[2]} = e_{2^n-2}$ (cf. [3, §1]). Thus $\mathfrak{H} = \mathfrak{W}$, and this yields the claim. \square

3.1. Toral complements. We define the restricted Lie subalgebra \mathfrak{B} of $\mathfrak{W}(1; n)$ by

$$(3.9) \quad \mathfrak{B} = \text{span}_{\mathbb{F}}\{e_j \mid 0 \leq j \leq p^n - 2\},$$

i.e., $\mathfrak{B} \subseteq W(1; n)$.

Proposition 3.2. *Let \mathbb{F} be a perfect field of characteristic p .*

- (a) *Let $\mathfrak{t} \subseteq \mathfrak{W}(1; n)$ be any torus of maximal dimension. Then \mathfrak{t} has dimension n .*
- (b) *There exists a torus $\mathfrak{t}_0 \subset \mathfrak{W}(1; n)$ of maximal dimension such that*

$$(3.10) \quad \mathfrak{W}(1; n) = \mathfrak{t}_0 \oplus \mathfrak{B}.$$

Proof. (a) For $p > 2$ the assertion has been shown in [6, Theorem 7.6.3]. Hence we may assume that \mathbb{F} is a perfect field of characteristic 2. Define $s = e_{-1} + e_{2^n-2} \in W(1; n)$. By induction, one concludes that

$$(3.11) \quad s^{2^k} = \partial^{2^k} + e_{2^n-2^k-1} \quad \text{for } k \in \{1, \dots, n-1\}, \text{ and}$$

$$(3.12) \quad s^{2^n} = \partial^{2^n} + e_{-1} + e_{2^n-2} = s.$$

In particular, the minimal polynomial of $\text{ad}_W(s)$ divides the separable polynomial $T^{2^n} - T \in \mathbb{F}[T]$. Thus $\text{ad}_W(s) \in \mathfrak{W}(1; n)$ is semi-simple and the elements $\text{ad}_W(s)^{2^i} \in \text{Der}(W(1; n)) = \mathfrak{W}(1; n)$, $0 \leq i \leq n-1$, are linearly independent. Hence $\mathfrak{t}_0 = \text{span}_{\mathbb{F}}\{\text{ad}(s)^{2^i} \mid 0 \leq i \leq n-1\}$ is a torus of $\mathfrak{W}(1; n)$ of dimension n , i.e., the maximal dimension $\text{MT}(\mathfrak{W}(1; n))$ of a torus of $\mathfrak{W}(1; n)$ must be greater or equal to n . Moreover the \mathbb{F} -algebras $\mathcal{O}(1; n)$ and $\mathcal{O}(n; \underline{1})$ are isomorphic, so there exists an injective homomorphism of restricted Lie algebras $i: \mathfrak{W}(1; n) \hookrightarrow \mathfrak{W}(n; \underline{1})$. Since $C(\mathfrak{W}(1; n)) = 0$ (cf. Proposition 3.1 and [3, Proposition 1.4]), it follows from [6, Lemma 1.2.6(1)] that $\text{MT}(\mathfrak{W}(1; n)) = \text{TR}(W(1; n))$, and thanks to [6, Theorem 1.2.7(1)] this yields $\text{MT}(\mathfrak{W}(1; n)) \leq \text{MT}(\mathfrak{W}(n; \underline{1})) = n$ (cf. [6, Corollary 7.5.2]). So we may conclude that the maximal dimension $\text{MT}(\mathfrak{W}(1; n))$ of a torus of $\mathfrak{W}(1; n)$ is equal to n .

(b) For $p = 2$ the just mentioned argument shows that $\mathfrak{t}_0 \oplus \mathfrak{B} = \mathfrak{W}(1; n)$. For $p > 2$ one may identify $W(1; n)$ with $L(\mathbb{E})$, where $\mathbb{E} \subseteq \mathbb{F}$, $|\mathbb{E}| = p^n$, and define $\mathfrak{t}_0 = \text{span}_{\mathbb{F}}\{u_0^{[p]^j} \mid 0 \leq j \leq n-1\}$ (cf. the proof of [6, Theorem 7.6.3(2)]). \square

3.2. The restricted subalgebra \mathfrak{B} . Note that $e_0 \in \mathfrak{B}$ is a toral element. Moreover, as $\text{rad}_u(\mathfrak{B}) = \text{span}_{\mathbb{F}}\{e_j \mid 1 \leq j \leq p^n - 2\}$, one has

$$(3.13) \quad \mathfrak{B} = \mathfrak{s} \oplus \text{rad}_u(\mathfrak{B}),$$

where $\mathfrak{s} = \mathbb{F} \cdot e_0$. For $\lambda \in \{0, \dots, p-1\}$ we denote by $\mathbb{F}[\lambda]$ the 1-dimensional restricted \mathfrak{B} -module satisfying

$$(3.14) \quad e_0 \cdot z = \lambda \cdot z \quad \text{for } z \in \mathbb{F}[\lambda].$$

In particular, $x \cdot z = 0$ for all $x \in \text{rad}_u(\mathfrak{B})$ and $z \in \mathbb{F}[\lambda]$. For our purpose the following property will turn out to be useful (cf. [3, Proposition 4.7(1) and (2)]).

Proposition 3.3. *Let \mathbb{F} be a field of characteristic $p > 0$ and let \mathfrak{B} be as described above. Then one has isomorphisms of restricted $\mathfrak{W}(1; n)$ -modules*

$$(3.15) \quad \text{ind}_{\mathfrak{B}}^{\mathfrak{W}(1; n)}(\mathbb{F}[-2]) \simeq W(1; n),$$

$$(3.16) \quad \text{ind}_{\mathfrak{B}}^{\mathfrak{W}(1; n)}(\mathbb{F}[1]) \simeq W(1; n)^* = \text{Hom}_{\mathbb{F}}(W(1; n), \mathbb{F}).$$

Remark 3.4. By Fact 2.1(a), $W(1; n)$ is a $\mathfrak{W}(1; n)$ -submodule of the adjoint $\mathfrak{W}(1; n)$ -module containing $L = \text{soc}_{\mathfrak{W}(1; n)}(\mathfrak{W}(1; n))$. Hence L is equal to $\text{soc}_{\mathfrak{W}(1; n)}(W(1; n))$. As $\dim(W(1; n)/L)$ is equal to 1 for $p = 2$ and 0 otherwise, this shows that L is the unique maximal $\mathfrak{W}(1; n)$ -submodule of $W(1; n)$.

3.3. Projective indecomposable modules. Proposition 3.2 has the following consequence for $p = 2$.

Theorem 3.5. *Let \mathbb{F} be an algebraically closed field of characteristic 2, let $n \geq 1$.*

- (a) $\text{Irr}_p(\mathfrak{W}(1; n)) = \{[\mathbb{F}], [L]\}$, where \mathbb{F} denotes the 1-dimensional trivial $\mathfrak{W}(1; n)$ -module, and $L = \text{soc}_{\mathfrak{W}(1; n)}(\mathfrak{W}(1; n)) = W(1; n)^{(1)}$ (cf. Remark 3.4).
- (b) If $n \geq 2$, \mathfrak{B} is a generalized Borel subalgebra of $\mathfrak{W}(1; n)$.
- (c) Let \mathfrak{t}_0 be a torus of maximal dimension of $\mathfrak{W}(1; n)$ satisfying $\mathfrak{W}(1; n) = \mathfrak{t}_0 \oplus \mathfrak{B}$ (cf. (3.10)). Then $L^{\mathfrak{t}_0} = 0$. In particular, $\mathfrak{W}(1; n)$ is a restricted Lie algebra with maximal 0-p.i.m., i.e., if $P_{\mathbb{F}}$ is the projective cover of the 1-dimensional trivial $\mathfrak{W}(1; n)$ -module, then one has $P_{\mathbb{F}} \simeq \text{ind}_{\mathfrak{t}_0}^{\mathfrak{W}(1; n)}(\mathbb{F}(0))$. Hence

$$(3.17) \quad \dim(P_{\mathbb{F}}) = 2^{\dim(\mathfrak{W}(1; n)) - \dim(\mathfrak{t}_0)} = 2^{2^n - 1}.$$

- (d) Let P_L denote the projective cover of the irreducible $\mathfrak{W}(1; n)$ -module L . Then $\dim(P_L) = 2^{2^n - 1}$. In particular, $P_L \simeq \text{ind}_{\mathfrak{t}_0}^{\mathfrak{W}(1; n)}(\mathbb{F}(\mu))$ for any non-trivial irreducible \mathfrak{t}_0 -module $\mathbb{F}(\mu)$, $\mu \in \mathfrak{t}_0^{\otimes} \setminus \{0\}$.

Proof. (a) Let $\mathfrak{W} = \mathfrak{W}(1; n)$. Since $\mathfrak{B}/\text{rad}_u(\mathfrak{B}) \simeq \mathfrak{s}$, one has

$$(3.18) \quad \text{Irr}_p(\mathfrak{B}) = \{[\mathbb{F}[0]], [\mathbb{F}[1]]\}.$$

Let S be an irreducible restricted \mathfrak{W} -module, and let $\Sigma = \text{soc}_{\mathfrak{B}}(\text{res}_{\mathfrak{B}}^{\mathfrak{W}}(S))$. Then Σ contains either a restricted \mathfrak{B} -submodule isomorphic to $\mathbb{F}[0]$ or $\mathbb{F}[1]$ or both. As

$$(3.19) \quad \text{Hom}_{\mathfrak{B}}(\mathbb{F}[\lambda], \text{res}_{\mathfrak{B}}^{\mathfrak{W}}(S)) \simeq \text{Hom}_{\mathfrak{W}}(\text{ind}_{\mathfrak{B}}^{\mathfrak{W}}(\mathbb{F}[\lambda]), S), \quad \lambda \in \{0, 1\},$$

Proposition 3.3 implies that S is either a homomorphic image of $\text{ind}_{\mathfrak{B}}^{\mathfrak{W}}(\mathbb{F}[0])$, in which case $S \simeq W/L \simeq \mathbb{F}$ (cf. Remark 3.4), or S is a homomorphic image of $\text{ind}_{\mathfrak{B}}^{\mathfrak{W}}(\mathbb{F}[1])$, in which case $S \simeq L^*$. Hence $\text{Irr}_p(\mathfrak{W}) = \{[\mathbb{F}], [L^*]\}$. As L is an irreducible restricted \mathfrak{W} -module, comparing dimensions yields that $L \simeq L^*$ as restricted \mathfrak{W} -modules.

(b) As $\mathfrak{B}/\text{rad}_u(\mathfrak{B}) \simeq \mathfrak{s}$ is a torus, $M^{\text{rad}_u(\mathfrak{B})}$ is a semi-simple restricted $\mathfrak{B}/\text{rad}_u(\mathfrak{B})$ -module for any finite-dimensional \mathfrak{W} -module M . Hence

$$(3.20) \quad M^{\text{rad}_u(\mathfrak{B})} = \text{soc}_{\mathfrak{B}}(\text{res}_{\mathfrak{B}}^{\mathfrak{W}}(M)).$$

Obviously, $\mathbb{F}^{\text{rad}_u(\mathfrak{B})} = \mathbb{F}[0]$ and $L^{\text{rad}_u(\mathfrak{B})} = \mathbb{F} \cdot e_{2^n-3}$, i.e., $L^{\text{rad}_u(\mathfrak{B})} \simeq \mathbb{F}[1]$ as \mathfrak{B} -module. So the mapping $\chi_{\mathfrak{B}} = [\text{rad}_u(\mathfrak{B})]: \text{Irr}_p(\mathfrak{W}) \rightarrow \text{Irr}_p(\mathfrak{B}/\text{rad}_u(\mathfrak{B}))$ is injective showing that \mathfrak{B} is a generalized Borel subalgebra.

(c) As $\mathfrak{W} = \mathfrak{t}_0 \oplus \mathfrak{B}$, one has

$$(3.21) \quad U = \text{res}_{\mathfrak{t}_0}^{\mathfrak{W}}(\text{ind}_{\mathfrak{B}}^{\mathfrak{W}}(\mathbb{F}[0])) \simeq \underline{u}(\mathfrak{t}_0) \otimes_{\mathbb{F}} \mathbb{F}[0] \simeq \underline{u}(\mathfrak{t}_0),$$

i.e., U is a free $\underline{u}(\mathfrak{t}_0)$ -module of rank 1. As $\text{ind}_{\mathfrak{B}}^{\mathfrak{W}}(\mathbb{F}[0])$ is isomorphic to the restricted \mathfrak{W} -module W (cf. (3.15)), one has a short exact sequence of restricted \mathfrak{t}_0 -modules

$$(3.22) \quad 0 \longrightarrow \text{res}_{\mathfrak{t}_0}^{\mathfrak{W}}(L) \longrightarrow U \longrightarrow \mathbb{F}(0) \longrightarrow 0,$$

where $\mathbb{F}(0)$ denotes the 1-dimensional trivial \mathfrak{t}_0 -module, i.e., $\text{res}_{\mathfrak{t}_0}^{\mathfrak{W}}(L)$ is isomorphic to the augmentation ideal $\ker(\varepsilon: \underline{u}(\mathfrak{t}_0) \rightarrow \mathbb{F})$ of $\underline{u}(\mathfrak{t}_0)$. Hence $L^{\mathfrak{t}_0} = 0$. Since any non-trivial, irreducible restricted \mathfrak{W} -module must be isomorphic to L , one concludes from [4, Lemma 6.1] that the projective cover $P_{\mathbb{F}}$ of the 1-dimensional irreducible \mathfrak{W} -module \mathbb{F} must be isomorphic to $\text{ind}_{\mathfrak{t}_0}^{\mathfrak{W}}(\mathbb{F}(0))$.

(d) As one has an isomorphism $\underline{u}(\mathfrak{W}) \simeq P_{\mathbb{F}} \oplus \dim(L) \cdot P_L$ of \mathfrak{W} -modules, one concludes that

$$(3.23) \quad \dim(P_L) = \frac{2^{2^n+n-1} - 2^{2^n-1}}{2^n - 1} = \frac{2^{2^n-1}(2^n - 1)}{2^n - 1} = 2^{2^n-1}.$$

If $\mathbb{F}(\mu)$ is a non-trivial irreducible \mathfrak{t}_0 -module, then $\text{ind}_{\mathfrak{t}_0}^{\mathfrak{W}}(\mathbb{F}(\mu))$ is projective, and has dimension equal to 2^{2^n-1} . As $\mathbb{F}(\mu)$ is isomorphic to a direct summand of L (cf. (3.19)), P_L is a homomorphic image of $\text{ind}_{\mathfrak{t}_0}^{\mathfrak{W}}(\mathbb{F}(\mu))$. Since both restricted \mathfrak{W} -modules have the same dimension, they must be isomorphic. This completes the proof. \square

Remark 3.6. In particular, Theorem 3.5(b) is also true for $n = 1$ if in the definition of a generalized Borel subalgebra condition (i) is omitted.

Remark 3.7. Let \mathbb{F} be an algebraically closed field of characteristic p , let \mathfrak{L} be a finite-dimensional restricted Lie algebra, and let $\mathfrak{t} \subseteq \mathfrak{L}$ be a torus of maximal dimension. If P is a projective indecomposable restricted \mathfrak{L} -module, then P is isomorphic to the projective cover P_S for some irreducible restricted \mathfrak{L} -module S . Let $\mathbb{F}(\mu)$ be an irreducible restricted \mathfrak{t} -submodule of $\text{res}_{\mathfrak{t}}^{\mathfrak{L}}(S)$. Then - as $\text{Hom}_{\mathfrak{L}}(\text{ind}_{\mathfrak{t}}^{\mathfrak{L}}(\mathbb{F}(\mu)), S) \neq 0$ - P is isomorphic to a direct summand of $P(\mu) = \text{ind}_{\mathfrak{t}}^{\mathfrak{L}}(\mathbb{F}(\mu))$ which is a projective restricted \mathfrak{L} -module for any $\mu \in \mathfrak{t}^{\otimes}$. Hence

$$(3.24) \quad \dim(P) \leq p^{\dim(\mathfrak{L}) - \text{MT}(\mathfrak{L})}.$$

This shows that for $\mathfrak{L} = \mathfrak{W}(1; n)$, $n > 1$, and $p = 2$, equality holds in (3.24) for any projective indecomposable restricted \mathfrak{L} -module P . Therefore, for the restricted Lie algebra \mathfrak{W} all p.i.m.s have maximal possible dimension.

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